$$\sigma_y(x) = \sigma_0 + O(\sqrt{b_1 - x}), \quad x \to b_1 - 0$$
$$\tau_{xy}(x) = K_{II}[2\pi(b_1 - x)]^{-1/2} + O(\sqrt{b_1 - x}), \quad x \to b_1 - 0$$

$$K_I = -\frac{\varkappa - 1}{\varkappa + 1} K_{II}, \quad K_{II} = \frac{(\varkappa + 1) F b_1 (\cos\theta - \omega \sin\theta)}{2\sqrt{\pi \varkappa b_1 (a_1^2 - b_1^2)}} \qquad (3.16)$$
$$\sigma_0 = \frac{(\varkappa + 1) F \sin\theta}{2\pi \sqrt{a_1^2 - b_1^2}}$$

Figure 2 shows how the dimensionless quantity $K_* = K_{II} F^{-1} \sqrt{2\pi a_1}$ varies as a function of b_1/a_1 in the case of a normal load ($\theta = {}^3/_2\pi$) for $\varkappa = 1,8$ and $\varkappa = 3$ (the dashed lines *1* and *2*). As expected, the tangential stress intensity coefficient K_{II} increases monotonically and without limit as the length of the separation segment increases. Thus, if the length of a closed crack reaches its critical value, it becomes globally unstable under a constant load and spreads over the whole segment $[-a_1, a_1]$.

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LYAPUNOV STABILITY AND SIGN DEFINITENESS OF A QUADRATIC FORM IN A CONE*

L.B. RAPOPORT

The use of the second Lyapunov method in many problems in the theory of stability of motion leads to the problem of sign definiteness of a quadratic form whose variables are defined in a convex polyhedral cone $C \subset R^n$. A method of obtaining the necessary and sufficient conditions is given for this problem. The conditions imposed on the elements of the third- and fourth-order matrices are given. The problem of asymptotic stability of a system with resonance /1/ is solved as an example.

A number of problems of the theory of the stability of motion require that the sign definiteness of the quadratic form be established, with conditions written in the form of linear inequalities. Usually, the conditions are those of non-negativity /1-3/, and the more general conditions can be reduced to them. The problem of sign definiteness of a quadratic form under the conditions of non-negativeness was considered for an arbitrary number of variables in /4/. However, the results obtained there can be reduced to the problem of the compatibility of systems of inequalities and pose well-known difficulties when used to solve specific problems.

The problem of the sign definiteness of a quadratic form in a convex cone

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(in general, infinitely-sided) belonging to a Hilbert space is considered in /5/, and the necessary and sufficient conditions are obtained, but in the finite-dimensional case discussed below the above result is the same as that obtained in /4/.

1. Let us consdier the problem of the sign definiteness of a real quadratic form in an arbitrary, polyhedral cone

$$C = \left\{x \in R^n \,\middle|\, x = \sum_{i=1}^{k} y_i r^i,\ y_i \geqslant 0,\ i = 1, \ldots k\right\}, \quad C \subset R^n \tag{1.1}$$

The real, symmetric $n \times n$-matrix A with components $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, n)$ will be called conditionally positive in the cone C, provided that the quadratic form $x^T Ax$ satisfies the codition

$$x^T Ax > 0, \quad x \in C \tag{1.2}$$

(we will assume the vectors to be column vectors and T denotes transposition).

In accordance with the representation (1.1), problem (1.2) reduces to the corresponding problem for the quadratic form

$$\Big(\sum_{i=1}^{k} y_i r^i\Big)^T A \Big(\sum_{j=1}^{k} y_j r^j\Big)$$

in the variables y_i, under the conditions that $y_i \geqslant 0$ $(i = 1, \ldots, k)$. It is sufficient therefore to consider the problem of the conditions of positiveness (CP) of the matrix in a non-negative cone. Below we shall consider only this problem, denoting everywhere by

$$R_+^n = \{x \in R^n \,|\, x_i \geqslant 0,\ i = 1, \ldots, n\}$$

the non-negative cone in the space R^n.

The method of analysing the CP of the matrix in the non-negative cone is based on the n-dimensional induction. According to the inductive assumption the quadratic form is positive on the $(n-1)$-dimensional sides of the cone R_+^n, and this is obviously necessary for the CP of the matrix A in R_+^n.

Let us denote by $A^{(i_1, i_2, \ldots, i_m)}$ the $(n-m) \times (n-m)$-matrix obtained from A by deleting the rows and columns with the indices $i_1 < i_2 < \ldots < i_m$. The positiveness of the quadratic form on the sides of the cone R_+^n is equivalent to the CP of the matrices $A^{(i)}$ and R_+^{n-1} when $i = 1, \ldots, n$.

Let us establish the conditions under which the inductive assumption is not only necessary, but also sufficient. Let us denote by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ the non-increasing sequence of eigenvalues of the matrix A, and by $q^1, q^2, \ldots, q^n$ the orthonormed system of the eigenvectors; $p(A)$ is the number of positive eigenvalues.

Lemma 1. If $p(A) \leqslant n-1$, then the necessary and sufficient condition for the CP of the matrix A in R_+^n is that the matrix $A^{(i)}$ $(i = 1., \ldots, n)$ be conditionally positive in R_+^{n-1} and some vector x^* for which

$$(x^*)^T Ax^* \leqslant 0 \tag{1.3}$$

has components of differing sign.

Proof. The matrices $A^{(i)}$ are necessarily conditionally positive. If the vector x^* has no components of different sign, then either $x^* \in R_+^n$, or $-x^* \in R_+^n$, which by virtue of (1.3) contradicts the CP of the matrix A in R_+^n.

Let us prove the sufficiency. We will assume that the assertion of the lemma is false. Then a vector $y \in R_+^n$ can be found such that $y^T Ay \leqslant 0$, and by virtue of the CP of the matrix $A^{(i)}$ in R_+^{n-1} we have

$$y_i > 0, \ i = 1, \ldots, n \tag{1.4}$$

(the equality $y_k = 0$ contradicts the CP of the matrix $A^{(k)}$). Having constructed for the scalar τ from the segment $[0, 1]$ the vector $x(\tau) = \tau x^* + (1-\tau) y$, we obtain

$$(x(\tau))^T Ax(\tau) = \tau^2 (x^*)^T Ax^* + 2\tau(1-\tau) y^T Ax^* + (1-\tau)^2 y^T Ay \tag{1.5}$$

Replacing, if necessary, x^* by $-x^*$, we obtain $y^T Ax^* \leqslant 0$ and by virtue of (1.5) we have

$$(x(\tau))^T Ax(\tau) \leqslant 0 \tag{1.6}$$

for all $0 \leqslant \tau \leqslant 1$. For sufficiently small τ (1.4) yields $x_i(\tau) > 0, i = 1, \ldots, n$, and since the vector x^* has components of different sign, it follows that we have $x(\tau_0) \in R_+^n$ at some $0 < \tau_0 < 1$ and $x_k(\tau_0) = 0$ for some k. This, together with (1.6), contradicts the CP of the matrix $A^{(k)}$ in R_+^{n-1}.

Below we shall show that when $p(A) < n-1$, the inductive assumption becomes sufficient without additional conditions.

Lemma 2. If $p(A) < n-1$, then the necessary and sufficient condition for CP of the matrix A in R_+^n, is the CP of the matrix $A^{(i)}$ $(i = 1, \ldots, n)$ in R_+^{n-1}.

Proof. Here we only need to show the sufficiency. Let us take any vector y with positive

components
$$y_i > 0,\ i = 1, \ldots, n \tag{1.7}$$

Since $p(A) + 1 < n$, it follows that we can find for $p(A) + 1$ vectors $y, q^1, \ldots, q^{p(A)}$ a vector $x^* \neq 0$ orthogonal to them
$$(x^*)^T y = 0, \quad (x^*)^T q^i = 0, \quad i = 1, \ldots, p(A) \tag{1.8}$$

Then, using the spectral expansion /6/ of the matrix A, we obtain
$$(x^*)^T Ax^* = \sum_{i=1}^{n} \lambda_i ((x^*)^T q^i)^2 = \sum_{i=p(A)+1}^{n} \lambda_i ((x^*)^T q^i)^2 \leqslant 0 \tag{1.9}$$

Conditions (1.7) and the first equation of (1.8) together imply that the vector x^* has components of different sign. This, together with (1.9), and by virtue of Lemma 1, means that CP of the matrix A in R_+^n holds.

Thus, when $p(A) < n$, the matrix A is conditionally positive in R_+^n if and only if all matrices $A^{(i)}, i = 1, \ldots, n$ are conditionally positive in R_+^{n-1}, and in the case of $p(A) = n - 1$ any vector chosen in accordance with condition (1.3), has components of different sign. Let us discuss the geometrical meaning of this result. We denote by
$$\Omega = \{x \in R^n \mid x^T A x \leqslant 0\} \setminus \{0\} \tag{1.10}$$
the region of the space R^n containing the vectors which impart positive values to the quadratic form, except for the null vector. We shall also specify, together with R_+^n, the cone R_-^n by means of the condition that $x \in R_-^n$ if $-x \in R_+^n$. Lemma 2 means that if the region Ω does not intersect the boundary of the cone R_+^n (and R_-^n since $(-x)^T A(-x) = x^T Ax$), then Ω has no common points with the inner part $\operatorname{int}(R_+^n \cup R_-^n)$ of these cones either. Indeed, when $p(A) < n - 1$, at least two eigenvectors corresponding to the non-positive eigenvalues of the matrix A belong to Ω together with the whole subspace (except for the point 0) stretched over them. But the subspace cannot have common points with the inner part of the cones R_+^n and R_-^n without intersecting their boundaries. If on the other hand $p(A) = n - 1$, then the region Ω will consists of two subregions Ω_1 and Ω_2, each represented by a sharp cone with the apex at the point 0, and if $x \in \Omega_1$, then $-x \in \Omega_2$ when $\lambda_n = 0\ \Omega_1$ and Ω_2 are semistraight lines. Now it is possible that $\Omega \subset \operatorname{int}(R_+^n \cup R_-^n)$. Applying Lemma 1 to this case we find, that if Ω does not intersect the boundary $R_+^n \cup R_-^n$, then $\operatorname{int}(R_+^n \cup R_-^n)$ together with any single vector $x^* \in \Omega$ either contains the whole region Ω as well, or it does not.

2. Thus when $p(A) \leqslant n - 1$, we must check the CP of all matrices $A^{(i)}$ in R_+^{n-1} and find, when $p(A) = n - 1$, a vector x^* which satisfies the condition (1.3) and check the signs of its components. When $p(A) = n$, the matrix A is positive definite and hence conditionally positive in R_+^n. According to Cauchy's theorem on separation /7/, the following estimate holds for the matrices $A^{(i)}\ (i = 1, \ldots, n)$:
$$p(A) - 1 \leqslant p(A^{(i)}) \leqslant p(A) \tag{2.1}$$

Therefore, if $p(A) = n - 1$, then either $p(A^{(i)}) = n - 1$ or $p(A^{(i)}) = n - 2$. In the first case the $(n-1) \times (n-1)$-matrix $A^{(i)}$ is positive definite. In the second case we must apply Lemma 1 to the matrix $A^{(i)}$. If the vector $x^* \in R^{n-1}$ satisfies the conditions of Lemma 1 for the matrix $A^{(i)}$, then the vector $x^{**} \in R^n$ which complements x^* with the component $x_i^{**} = 0$ will satisfy the conditions of Lemma 1 for the matrix A. Therefore, when analysing the CP of matrix A, we must consider the vector x^* only in the case when $p(A) = n - 1$ and $p(A^{(i)}) = n - 1$ for all $(i = 1, \ldots, n)$, i.e. when all matrices $A^{(i)}$ are positive definite. Such matrices (we shall call them minimal) satisfy the following conditions: $\det A \leqslant 0$ and $A^{(i)}$ are positive definite $i = 1, \ldots, n$. It is clear that n matrices $A^{(i)}$ will be positive definite if and only if one of them, e.g. $A^{(n)}$, in positive definite (i.e. has positive angular minors) and the rest satisfy the condition $\det A^{(i)} > 0$. Thus the conditions for the matrix A to be minimal take the form
$$a_{11} = \det A^{(2, \ldots, n)} > 0, \quad \det A^{(3, \ldots, n)} > 0, \ \ldots, \ \det A^{(n)} > 0 \tag{2.2}$$
$$\det A^{(i)} > 0, \ i = 1, \ldots, n - 1 \tag{2.3}$$
$$\det A \leqslant 0 \tag{2.4}$$
where $\det A^{(k, \ldots, n)}$ are the angular minors of the matrix $A^{(n)}$ consisting of $k - 1$ rows and columns.

Let us denote by A_{ij} the cofactors of the elements a_{ij} in the minimal matrix A, and by $C^i \in R^n$ a vector of the form $(A_{i1}, \ldots, A_{in})^T$. Then, since $C_i^i = A_{ii} = \det A^{(i)} > 0$, we have, by virtue of the properties of the cofactors,
$$(C^i)^T AC^i = \det A^{(i)} \det A \leqslant 0$$

Therefore we can take ony of the vectors $C^i\ (i = 1, \ldots, n)$ as the vector x^* satisfying condition (1.3). From Lemma 1 it follows that if the matrix A is minimal, then either every vector $C^i\ (i = 1, \ldots, n)$ has a negative component, or all vectors C^i are non-negative. To be specific, we shall take C^n as x^*.

The above discussion leads to the following formulation of the final result.

Theorem 1. In order for the matrix A to be positive in R_+^n, it is necessary and sufficient that one of the following three conditions holds: 1) the matrix A is positive definite, 2) all matrices $A^{(i)}$ are conditionally positive in R_+^{n-1} and at least one of these matrices is positive definite, 3) the matrix A is minimal and at least one of the numbers $A_{n1}, A_{n2}, \ldots, A_{n,n-1}$ is negative.

We use the above theorem in Sect.3 to obtain the algebraic criterion for the CP.

Let us obtain a corollary to Theorem 1, which is interesting in itself. We shall call the matrices of the form $A^{(i_1,\ldots,i_m)}$ the submatrices of A. From the definition of the minimal matrix and the Cauchy theorem on the distribution, it follows that if $1 \leqslant p(A) \leqslant n-1$, then the matrix A will contain minimal submatrices. The condition $p(A) \geqslant 1$ holds if the matrix A has positive diagonal elements, and this is necessary for the CP. Applying Theorem 1 by induction, we obtain

Corollary. The necessary and sufficient condition for the matrix A to be positive is that its diagonal elements are positive and one of the following two conditions holds: 1) the matrix A is positive definite, 2) all minimal submatrices of A have, amongst the cofactors of their last (to be specific) rows, at least one negative component.

3. The criterion given below reduces the problem of CP to that of analysing the signs of the minors of A, i.e. it represents the set of conditions for the elements of the matrix.

Theorem 2. The necessary and sufficient condition for the matrix A in R_+^n to be positive is the CP of all matrices $A^{(i)}$ $(i=1,\ldots,n)$ in R_+^{n-1} and, that at least one of the following conditions holds:

$$\det A^{(3,\ldots,n)} \leqslant 0, \ldots, \det A^{(n-1,n)} \leqslant 0, \quad \det A^{(n)} \leqslant 0 \tag{3.1}$$

$$\det A^{(1)} \leqslant 0, \ldots, \det A^{(n-1)} \leqslant 0 \tag{3.2}$$

$$\det A > 0 \tag{3.3}$$

$$A_{n1} < 0, \ldots, A_{n,n-1} < 0 \tag{3.4}$$

(The conditions listed in (3.1) and (3.2) hold when $n \geqslant 3$.)

Proof. We shall establish the equivalence between the conditions of this theorem, and those of Theorem 1. Let condition 1) of Theorem 1 hold. Then all matrices $A^{(1)}, \ldots, A^{(n)}$ will be positive definite and condition (3.3) will hold. In this case the conditions of Theorem 2 will also hold. If condition 2) of Theorem 1 holds but conditions (2.2)-(2.4) do not hold, then at least one of the conditions listed in (3.1)-(3.3) will hold. The first condition of (2.2) $(a_{11} > 0)$ must hold, since it is necessary for the CP of A in R_+^n. Moreover, $A^{(1)}, \ldots, A^{(n)}$ are conditionally positive and the conditions of Theorem 2 again hold. In case 3) of Theorem 1 the conditions (2.2)-(2.4) and at least one of the conditions (3.4) all hold. Since all matrices $A^{(1)}, \ldots, A^{(n)}$ of the minimal matrix A are positive definite, it follows that the conditions of Theorem 2 hold in this case also.

Next we shall show that one of the conditions of Theorem 1 follows from the conditions of Theorem 2. If none of the conditions listed in (3.1)-(3.2) hold, then all matrices $A^{(1)}, \ldots, A^{(n)}$ are positive definite (the CP of the matrix $A^{(1)}$ also implies that $a_{11} > 0$). If in addition (3.3) holds, then A is positive definite and condition 1) of Thoerem 1 is satisfied. If (3.3) does not hold, then according to (2.2)-(2.4) the matrix A is minimal. One of the remaining conditions of (3.4) holds by virtue of the conditions of the theorem, and condition 3) of Theorem 1 also holds. If on the other hand one of the conditions (3.1)-(3.2) holds, then at least one of the matrices $A^{(1)}, \ldots, A^{(n)}$ is not positive definite and condition 2) of Theorem 1 is satisfied. The theorem is proved.

The above theorem, applied by induction, enables us to obtain a set of conditions in the form of inequalities for the matrix coefficients connected by the symbols AND and OR, for any dimensionality n.

4. Let us consider the cases when $n = 2, 3, 4$, which are most often encountered in problems of stability.

$n = 2$. This elementary case is of interest, since it represents the first induction step when Theorem 2 is used. The matrices $A^{(1)}$ and $A^{(2)}$ each have a single element a_{22} and a_{11}, the cofactor $A_{21} = -a_{12}$, and the CP of $A^{(1)}$ and $A^{(2)}$ mean that

$$a_{11} > 0, \ a_{22} > 0 \tag{4.1}$$

Not a single condition of (3.1), (3.2) holds, and conditions (3.3), (3.4) yield $\det A > 0$, $a_{12} > 0$. When at least one of these inequalities is satisfied, this is equivalent to the condition

$$\max\{\det A, a_{12}\} > 0 \tag{4.2}$$

which agrees with the well-known criterion (see e.g. /4/).

$n = 3$. In this case we have

$$A^{(1)}=\left\|\begin{matrix}a_{22} & a_{23}\\ a_{23} & a_{33}\end{matrix}\right\|,\quad A^{(2)}=\left\|\begin{matrix}a_{11} & a_{13}\\ a_{13} & a_{33}\end{matrix}\right\|,\quad A^{(3)}=\left\|\begin{matrix}a_{11} & a_{12}\\ a_{12} & a_{22}\end{matrix}\right\|$$

$$A_{31}=\left|\begin{matrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{matrix}\right|,\quad A_{32}=-\left|\begin{matrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{matrix}\right|$$

Applying Theorem 2, we obtain the following conditions. The matrices $A^{(1)}, A^{(2)}, A^{(3)}$ must be conditionally positive, and this means, taking into account (4.1) and (4.2), that the following conditions hold:

$$a_{11}>0,\ a_{22}>0,\ a_{33}>0,\ \max\{\det A^{(1)},\ a_{23}\}>0 \tag{4.3}$$
$$\max\{\det A^{(2)},\ a_{13}\}>0,\ \max\{\det A^{(3)},\ a_{12}\}>0$$

According to (3.1)-(3.4), the CP of matrix A requires, in addition to (4.3), that one of the following conditions be satisfied:

$$\det A^{(1)}\leqslant 0,\quad \det A^{(2)}\leqslant 0,\quad \det A^{(3)}\leqslant 0 \tag{4.4}$$
$$\det A>0,\quad \left|\begin{matrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{matrix}\right|<0,\quad \left|\begin{matrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{matrix}\right|>0$$

$n=4$. The conditions for the third-order matrices $A^{(i)}\ (i=1,\ldots,4)$ are given above. According to Theorem 2, in addition to these conditions at least one of the inequalities obtained from (3.1)-(3.4) and taking the form

$$\det A^{(3,4)}=\left|\begin{matrix}a_{11} & a_{12}\\ a_{12} & a_{22}\end{matrix}\right|\leqslant 0,\quad \det A^{(4)}\leqslant 0$$
$$\det A^{(1)}\leqslant 0,\quad \det A^{(2)}\leqslant 0,\quad \det A^{(3)}\leqslant 0,\quad \det A>0$$
$$\left|\begin{matrix}a_{12} & a_{13} & a_{14}\\ a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\end{matrix}\right|>0,\quad \left|\begin{matrix}a_{11} & a_{13} & a_{14}\\ a_{21} & a_{23} & a_{24}\\ a_{31} & a_{33} & a_{34}\end{matrix}\right|<0,\quad \left|\begin{matrix}a_{11} & a_{12} & a_{14}\\ a_{21} & a_{22} & a_{24}\\ a_{31} & a_{32} & a_{34}\end{matrix}\right|>0$$

must also hold.

5. The sufficient conditions for the asymptotic stability were constructed in /1/ for the trivial solution of the autonomous system of ordinary differential equations with the fourth-order resonance. The conditions were formulated as the conditions of negative definiteness of a matrix whose elements were obtained using the coefficients of normal form. It was noted in /1/ that the conditions can be broadened, since it is sufficient to demand the negative definiteness within the non-negative cone only. We consider the sign definiteness of the following matrix for the case of three degrees of freedom (the notation used in /1/ is retained here)

$$M=\left\|\begin{matrix}2\gamma_1 a_{11} & \gamma_1 a_{12}+\gamma_2 a_{21} & \gamma_1 a_{13}+\gamma_3 a_{31}\\ \gamma_1 a_{12}+\gamma_2 a_{21} & 2\gamma_2 a_{22} & \gamma_2 a_{23}+\gamma_3 a_{32}\\ \gamma_1 a_{13}+\gamma_3 a_{31} & \gamma_2 a_{23}+\gamma_3 a_{32} & 2\gamma_3 a_{33}\end{matrix}\right\|$$

where $\gamma_1=\gamma_3 D_{23}/D_{12}$, $\gamma_2=\gamma_3 D_{31}/D_{12}$, γ_3 is an arbitrary positive constant, D_{23}, D_{31}, D_{12} are the covariant components of the vector product $a\times b$, $a=(a_1, a_2, a_3)$, $b=(b_1, b_2, b_3)$, a_{ij} are the coefficients of the normal form (see (1.1) of /1/ and $D_{23}/D_{12}>0$ and $D_{31}/D_{12}>0$. Having written

$$\delta_1=\frac{D_{23}}{D_{12}}a_{12}+\frac{D_{31}}{D_{12}}a_{21},\quad \delta_2=\frac{D_{23}}{D_{12}}a_{13}+a_{31},\quad \delta_3=\frac{D_{31}}{D_{12}}a_{23}+a_{32}$$

and applying to the matrix M the condition (4.3), (4.4), we obtain

$$a_{11}<0,\ a_{22}<0,\ a_{33}<0 \tag{5.1}$$
$$\max\{d_{12}, -\delta_1\}>0,\ \max\ \{d_{13}, -\delta_2\}>0,\ \max\{d_{23}, -\delta_3\}>0 \tag{5.2}$$
$$d_{12}=4\frac{D_{23}D_{31}}{D_{12}^2}a_{11}a_{22}-\delta_1^2,\quad d_{13}=4\frac{D_{23}}{D_{12}}a_{11}a_{13}-\delta_2^2,\ d_{23}=4\frac{D_{31}}{D_{12}}a_{22}a_{33}-\delta_3^2$$

and at least one of the following conditions holds:

$$d_{12}\leqslant 0,\ d_{13}\leqslant 0,\ d_{23}\leqslant 0 \tag{5.3}$$
$$a_{33}d_{12}-\delta_3^2\frac{D_{23}}{D_{12}}a_{11}+\delta_1\delta_2\delta_3-\frac{D_{31}}{D_{12}}a_{22}\delta_2^2<0$$
$$\delta_1\delta_3-2\frac{D_{31}}{D_{12}}a_{22}\delta_2<0,\quad \delta_1\delta_2-2\frac{D_{23}}{D_{12}}a_{11}\delta_3<0$$

We see at once that conditions (5.1)-(5.3) widen conditions (3.2) of /1/.

In conclusion we note the paper by Molchanov /8/ where a similar algebraic problem occurs.

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THE USE OF THE METHOD OF AVERAGING TO STUDY NON-LINEAR OSCILLATIONS OF THE CELTIC STONE*

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The approximate solution of the equations of the perturbed motion of a celtic stone near its position of equilibrium, obtained in /1/ by retaining in these equations terms of the second order with respect to the perturbations and averaging** (**In connection with a footnote in /3/ which appeared later than /1/, we note that the solution obtained in /3/ is identical with that appearing in /1/) is used for a qualitative and quantitative explanation of the following effect established by numerical integration of the complete equations /2/. If the celtic stone is rotated about the vertical axis in a specified direction, then after a fairly short time it ceases to rotate, begins to oscillate about the horizontal axis, and then resumes its rotation in the opposite direction. For some of the models of the celtic stone the change in the direction of rotation may occur more than once.

Let m be the mass of the body, $Gx_1x_2x_3$ the coordinate system attached to the body whose axes are directed along the principal central axes of inertia of the body, A, B, C are the corresponding moments of inertia, x_0, y_0 are the horizontal coordinates of the centre of mass in the fixed coordinate system $O_0x_0y_0z_0$ (the $O_0x_0y_0$ plane is the same as the reference plane), φ, ψ, θ are the Euler angles determining the orientation of the system $Gx_1x_2x_3$ relative to $O_0x_0y_0z_0$, ξ, η, ζ are the coordinates of the point I of contact of the body with the reference plane in the system $Gx_1x_2x_3$, ρ_1, ρ_2 are the radii of curvature of the body at the point J with coordinates $(0, -a, 0)$ in the system $Gx_1x_2x_3$, α is the angle defining the position of the principal axes of curvature at the point J relative to the axes Gx_1, Gx_2. We study the same model of the celtic stone as that in /2/. In particular, the following inequalities hold for this model:

$$\rho_2 > \rho_1 > a, \quad 0 < \alpha < \pi/2, \quad ma\rho_1 < \bar{A} + \bar{C} - B < ma\rho_2$$
$$m\rho_1\rho_2 > B > \bar{A} > \bar{C} \quad (\bar{A} = A + ma^2, \ \bar{C} = C + ma^2)$$

The system under consideration represents a non-holonomic Chaplygin system. Its equations of motion are independent of the angle ψ and admit of the family of solutions

$$\theta = \pi/2, \ \varphi = 0, \ \psi^{\cdot} = \omega \tag{1}$$

where ω is an arbitrary constant. The solutions correspond to uniform rotation of the body about the vertical axis Gx_2, and to the equilibrium state of the body when $\omega = 0$. The point of contact I of the body with the plane coincides with the point J of the body.

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